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Exact propagator for a two-dimensional inverse quadratic oscillator interacting with a wedge

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Abstract. The exact propagator of our dynamic system is derived for a rational wedge. For an irrational wedge, the proposed propagator can be confirmed by expanding it in terms of eigenfunctions and eigenvalues, which agree with those obtained from the corresponding Schrödinger equation. Our results are also valid for an inverse square potential interacting with a wedge. Finally we investigate the classical path's contributions to the propagator for the free particle and the rational wedge case.

1. Introduction

Recently, the 2D dynamical systems interacting with a wedge have been studied quantum mechanically. On the one hand, Crandall [1] and DeWitt-Morette *et al* [2] evaluate the propagator for a free particle interacting with a rational wedge. On the other hand, Schulman [3] and Wiegel and van Andel [4] obtain, respectively, the exact propagator for a free particle and for a harmonic oscillator, both interacting with an infinite half-plane barrier. Their results have been generalized [5] by Cheng (*a*) by including a harmonic potential and (*b*) by extending to an irrational wedge [6].

In this paper, we first derive the propagator for a rational wedge case in section 2. For an irrational wedge case, we propose the propagator from which we then obtain the energy eigenfunctions and eigenvalues, respectively, in section 3. For comparison, we solve the corresponding Schrödinger equation and show that the propagator in section 3 is correct in section 4. In section 5, we discuss the classical path's contributions to the propagator for the free particle interacting with a rational wedge in general and with a half-plane barrier in particular.

2. Path integral evaluations

For purposes to be explained later, we first consider the dynamical system of the following Lagrangian:

$$L(r, r) = \left(\frac{\mu}{2}\right)[(x^2 + y^2) - \omega^2 r^2] + \frac{q\Phi(x\dot{y} - \dot{x}y)}{2\pi r^2} - \frac{g}{r^2} \tag{1}$$

where q and μ are the charge and the mass of the particle. Here we have assumed (i) a harmonic potential with the same angular frequence ω in x and y directions, (ii) an inverse square potential with strength $g > 0$, (iii) a vector potential of a long solenoid of zero size at the origin and containing the magnetic flux Φ [11]. Therefore the charged particle moves in a 2D multiply-connected space with the origin removed. Evaluating the path integral with the Lagrangian (1) we recently obtained the propagator in polar coordinates as [7-11]

$$K(r'',\varphi'',t;r',\varphi';f) = \left[\frac{\mu\omega}{2\hbar\pi \mathrm{i}\sin(\omega t)}\right]\exp[\mathrm{i}\mu\omega(r'^2+r''^2)\cot(\omega t)/2\hbar] \times \sum_{l=-\infty}^{\infty}\exp[-\mathrm{i}(l+f)(\varphi''-\varphi')](-\mathrm{i})^{|\lambda_f|}J_{|\lambda_f|}\left(\frac{\mu\omega r'r''}{\hbar\sin(\omega t)}\right) \tag{2}$$

with $\lambda_f^2=[(l+f)^2+2\mu g/\hbar^2]$. $J_{|\lambda_f|}(\cdot)$ is the Bessel function and $f=q\Phi/2\pi\hbar c$ is the flux quantum number of the solenoid.

For our dynamical system we assume that (i) the wedge is along the z-axis, (ii) the external angle of wedge is $a\pi$ $(0<a\leqslant 2)$, (iii) the harmonic potential is centred at the origin (see figure 1) and (iv) the inverse square potential is of the form g/r^2 $(g>0)$. However, we assume $a=n\pi/m$ is rational (n and m are positive integers), in this section. Now we have to evaluate the path integral on an n-sheeted Riemann surface since the paths during the time interval of t must loop a multiple of n times to go from the initial position (r',φ') to the final position (r'',φ''), which are both on the top (or physical) sheet. Using the linear combinations of the propagator (2) with $\omega=0$ and with different values of f (a multiple of $1/n$), DeWitt-Morette *et al* [2] are able to obtain a free propagator on an n-sheeted Riemann surface. Extending their result, we arrive at

$$K_{\mathrm{R}}(r'',\varphi'',t;r',\varphi') = \frac{1}{n}\left[\frac{\mu\omega}{2\hbar\pi \mathrm{i}\sin(\omega t)}\right]\exp[\mathrm{i}\mu\omega(r'^2+r''^2)\cot(\omega t)/2\hbar] \times \sum_{l=-\infty}^{\infty}(\exp[-\mathrm{i}l(\varphi''-\varphi')/n])(-\mathrm{i})^{|\nu_l|}J_{|\nu_l|}\left(\frac{\mu\omega r'r''}{\hbar\sin(\omega t)}\right) \tag{3}$$

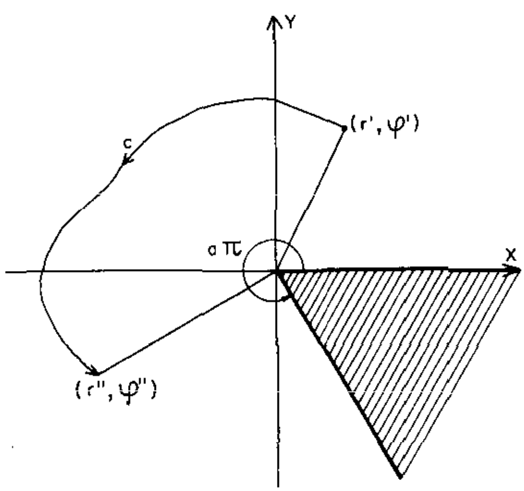

Figure 1. The edge of the wedge is the z-axis. The harmonic plus inverse square potential is centred at the origin of the (r,φ) plane. The path C represents one of the paths from the initial position (r',φ') to the final position (r'',φ'') during the time interval of t.

where $\nu_l^2 = (l^2/n^2 + 2\mu g/\hbar^2)$, which is the propagator of a 2D inverse square oscillator on an n-sheeted Riemann surface.

By considering the contributions from the source (particle) and its images reflecting with respect to the wedge surfaces, we can satisfy the boundary conditions we need on the surfaces of the wedge. However, the image which achieves the proper boundary conditions on one surface will disturb the boundary conditions on the other. Therefore, we should add all images of these images until the set of images closes on itself. Following the analysis of Dewitt-Morette *et al* [2], the even (odd) images obtained from the source by even (odd) number of reflections stand at $\varphi = \varphi' + 2\pi kn/m$ ($\varphi = -\varphi' + 2\pi kn/m$) for k as integer. There are $2m$ integers (including the source) in total on an n-sheeted Riemann surface.

In order to satisfy the Dirichlet boundary conditions in quantum mechanics, the propagator must vanish on the surfaces of the wedge, and it can be obtained by summing the probability amplitude of the even and odd image with opposite phase since they contribute the same strength of probability amplitude to the final position of the particle. We then have

$$K_{n/m}(r'', \varphi'', t; r', \varphi') = \frac{m}{n}\left[\frac{\mu\omega}{2\hbar\pi \mathrm{i}\sin(\omega t)}\right] \exp[\mathrm{i}\mu\omega(r'^2 + r''^2)\cot(\omega t)/2\hbar] \times \sum_{l=1}^{\infty} (-i)^{|\sigma_l|} \sin(lm\varphi''/n)\sin(lm\varphi'/n) J_{|\sigma_l|}\left(\frac{\mu\omega r' r''}{\hbar\sin(\omega t)}\right) \tag{4}$$

where $\sigma_l^2 = [(ml/n)^2 + 2\mu g/\hbar^2]$ for the propagator of the 2D inverse square oscillator interacting with a rational wedge.

3. Exact propagator for the irrational wedge case

We observe [6] that the propagator (4) can only be generalized for the irrational wedge case by replacing the factor n/m by a. Therefore, we propose the propagator as

$$K_a^{\omega}(r'', \varphi'', t; r', \varphi') = \left[\frac{\mu\omega}{2\hbar a\pi \mathrm{i}\sin(\omega t)}\right] \exp[\mathrm{i}\mu\omega(r'^2 + r''^2)\cot(\omega t)/2\hbar] \times \sum_{l=1}^{\infty} \sin(l\varphi''/a)\sin(l\varphi'/a) I_{\mu_l}\left(\frac{\mu\omega r' r''}{\mathrm{i}\hbar\sin(\omega t)}\right) \tag{5}$$

where $\mu_l^2 = [(l/a)^2 + 2\mu g/\hbar^2]$. To verify the validity of (5), we rewrite it as

$$\begin{aligned} K_a^{\omega}(r'', \varphi'', t; r', \varphi') &= \frac{4\mu\omega}{a\pi\hbar}\exp\left[-\frac{\mu\omega}{2\hbar}(r'^2 + r''^2)\right]\sum_{l=1}^{\infty}\sin(l\varphi''/a)\sin(l\varphi'/a) \\ &\times \exp(-\mathrm{i}\omega t)\left\{[1 - \exp(-2\mathrm{i}\omega t)]^{-1}\right. \\ &\times \exp\left[-\frac{\mu\omega}{\hbar}(r'^2 + r''^2)\frac{\exp(-2\mathrm{i}\omega t)}{1 - \exp(-2\mathrm{i}\omega t)}\right] \\ &\times \left. I_{\mu_l}\left(\frac{2\mu\omega}{\hbar} r' r'' \frac{\exp(-\mathrm{i}\omega t)}{1 - \exp(-2\mathrm{i}\omega t)}\right)\right\}. \end{aligned} \tag{6}$$

Using the identities [12]

$$(1-u)^{-1}\exp\left[-(c+d)\frac{u}{1-u}\right]I_\alpha\left(\frac{2(cdu)^{1/2}}{1-u}\right)$$
$$=(cdu)^{\alpha/2}\sum_{q=0}^{\infty}\frac{q!}{\Gamma(l/a+1+q)}L_q^\alpha(c)L_q^\alpha(d)u^q \qquad (|u|<1) \tag{7}$$

and

$$L_q^\alpha(v)=\binom{q+\alpha}{q}\,{}_1F_1(-q,\alpha+1;v) \tag{8}$$

let $u=\exp(-2\mathrm{i}\omega t)$, $c=\mu\omega r'^2/\hbar$, $d=\mu\omega r''^2/\hbar$, $\alpha=\mu_l$ and $q=n_r$, in (7), equation (6) then becomes

$$K_a^\omega(r'',\varphi'',t;r',\varphi')$$
$$=\frac{4\mu\omega}{a\pi\hbar}\exp\left[-\frac{\mu\omega}{2\hbar}(r'^2+r''^2)\right]\sum_{l=1}^{\infty}\sum_{n_r=0}^{\infty}\left[\binom{n_r+\mu_l}{n_r}\right]^2$$
$$\times\frac{n_r!}{\Gamma(\mu_l+1+n_r)}\left(\frac{\mu\omega}{\hbar}r'^2\right)^{\mu_l/2}\left(\frac{\mu\omega}{\hbar}r'^2\right)^{\mu_l/2}$$
$$\times\sin(l\varphi''/a)\sin(l\varphi'/a)\,{}_1F_1\left(-n_r,\mu_l+1;\frac{\mu\omega}{\hbar}r'^2\right)$$
$$\times\,{}_1F_1\left(-n_r,\mu_l+1;\frac{\mu\omega}{\hbar}r''^2\right)\exp[-\mathrm{i}(\mu_l+1+2n_r)\omega t]. \tag{9}$$

Comparing with the expression of the propagator in terms of the energy eigenvalues and eigenfunctions, we have

$$E_{n_r}=\hbar\omega(\mu_l+1+2n_r) \tag{10}$$

$$\psi_{n_r,l}(r,\varphi)=C_{n_r,l}r^{\mu_l}\exp(-\mu\omega r^2/2\hbar)\sin(l\varphi/a)\,{}_1F_1(-n_r,\mu_l+1;\mu\omega r^2/\hbar) \tag{11}$$

($n_r=0, 1, 2, \ldots$), with the normalization constant being

$$C_{n_r,l}=2\left[\frac{(\mu\omega/\hbar)^{\mu_l+1}n_r!}{a\pi\Gamma(\mu_l+1+n_r)}\right]^{1/2}\binom{n_r+\mu_l}{n_r}. \tag{12}$$

For the case of $\omega=0$ (without the harmonic potential), (5) reduces to

$$K_a^0(r'',\varphi'',t;r',\varphi')$$
$$=\left(\frac{2\mu}{a\pi\mathrm{i}\hbar t}\right)\exp[\mathrm{i}\mu(r'^2+r''^2)/2\hbar t]$$
$$\times\sum_{l=1}^{\infty}\sin(l\varphi'/a)\sin(l\varphi''/a)I_{\mu_l}\left(\frac{\mu r'r''}{\mathrm{i}\hbar t}\right). \tag{13}$$

Using the result [12]

$$\int_0^\infty\exp(-\alpha x)J_\nu(2\beta\sqrt{x})J_\nu(2\gamma\sqrt{x})\,\mathrm{d}x=\frac{1}{\alpha}I_\nu\left(\frac{2\beta\gamma}{\alpha}\right)\exp\left(-\frac{\beta^2+\gamma^2}{\alpha}\right). \tag{14}$$

If we let $\alpha = \mathrm{i}\hbar t/\mu$, $\beta = r'/\sqrt{2}$ and $\gamma = r''/\sqrt{2}$, then we have

$$\frac{\mu}{\mathrm{i}\hbar t}\exp\left[\frac{\mathrm{i}\mu}{2\hbar t}(r'^2+r''^2)\right]I_{\mu_l}\left(\frac{\mu r' r''}{\hbar \mathrm{i} t}\right) = \int_0^\infty \exp\left(-\frac{\mathrm{i}\hbar t}{\mu}y\right)J_{\mu_l}\left(\frac{2r'\sqrt{y}}{\sqrt{2}}\right)J_{\mu_l}\left(\frac{2r''\sqrt{y}}{\sqrt{2}}\right)\mathrm{d}y. \tag{15}$$

Now (13) can be put into the form

$$K_a^0(r'', \varphi'', t; r', \varphi') = \frac{2\mu}{a\pi\hbar^2}\sum_{l=1}^{\infty}\sin(l\varphi'/a)\sin(l\varphi''/a)\int_0^\infty \exp\left(-\frac{\mathrm{i}}{\hbar}Et\right) \times J_{\mu_l}(kr')J_{\mu_l}(kr'')\,\mathrm{d}E. \tag{16}$$

Therefore we get the energy eigenfunctions as

$$\psi_{l,E}(r,\varphi,t) = \left(\frac{2\mu}{a\pi\hbar^2}\right)^{1/2}\sin(l\varphi/a)J_{\mu_l}(kr)\exp(-\mathrm{i}Et/\hbar). \tag{17}$$

Writing the Bessel function as a degernate hypergeometric function (see table 9.238 − 1 in [12]), the exact propagator (5) reduces to

$$K_a^\omega(r'', \varphi'', t; r', \varphi') = \left[\frac{2\mu\omega}{a\pi\mathrm{i}\hbar\sin(\omega t)}\right]\exp\left\{\frac{\mathrm{i}\mu\omega}{2\hbar\sin(\omega t)}[(r'^2+r''^2) \times\cos(\omega t) - 2r'r'']\right\}\sum_{l=1}^{\infty}\frac{1}{\Gamma(\mu_l+1)}\left(\frac{\mu\omega r' r''}{2\mathrm{i}\hbar\sin(\omega t)}\right)^{\mu_l} \times {}_1F_1\left(\mu_l+\tfrac{1}{2}, 2\mu_l+1; \frac{2\mu\omega r' r''}{\mathrm{i}\hbar\sin(\omega t)}\right) \tag{18}$$

which does belong to the first group of the solvable propagator as classified by Inomata [13] as we expect. We should mention that for the $g=0$ case, (18) has been obtained by Cheng [6] and by Chetouani *et al* [14].

4. Schrödinger equation

4.1. Harmonic oscillator plus square potential

In polar coordinates, the Schrödinger equation is of the form

$$-\frac{\hbar^2}{2\mu}\left[\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\varphi^2}\right]\psi(r,\varphi)+\left[\frac{\mu\omega^2}{2}r^2+\frac{g}{r^2}\right]\psi(r,\varphi) = E\psi(r,\varphi). \tag{19}$$

The wavefunction must satisfy the following boundary conditons:

$$\psi(r,0) = \psi(r,a\pi) = 0. \tag{20}$$

We assume that the wavefunction has the form

$$\psi(r,\varphi) = R(r)Y(\varphi) \tag{21}$$

substituting (21) into (19), we find

$$Y(\varphi)=\sin(l\varphi/a) \qquad (l=1,2,3,\ldots) \tag{22}$$

in order to satisfy (20), and

$$R''+\frac{R'}{r}+\left[-\frac{\mu\omega}{\hbar}r^2-\frac{2\mu g}{(\hbar r)^2}+k^2\frac{(l/a)^2}{r^2}\right]R=0 \tag{23}$$

where $k^2=2\mu E/\hbar^2$. Assuming [15]

$$R(r)=r^{\mu_l}\exp(-\mu\omega r^2/2\hbar)F(r) \qquad \mu_l=[(l/a)^2+2\mu g/\hbar^2]^{1/2} \tag{24}$$

we have

$$\frac{d^2F(r)}{dr^2}+\left(\frac{2\mu_l+1}{r}-\frac{2\mu\omega}{\hbar}r\right)\frac{dF(r)}{dr}-\left[\frac{2\mu\omega}{\hbar}(\mu_l+1)-k^2\right]F(r)=0. \tag{25}$$

Using the variable $\xi=\mu\omega r^2/\hbar$, (25) reduces to the Kummar equation

$$\xi\frac{d^2F(\xi)}{d\xi^2}+[\mu_l+1-\xi]\frac{dF(\xi)}{d\xi}-\left[\tfrac{1}{2}(\mu_l+1)-\frac{\hbar k^2}{4\mu\omega}\right]F(\xi)=0 \tag{26}$$

whose solution regular at $r=0$ (or $\xi=0$) is the degenerate hypergeometric function

$$F(\mu\omega r^2/\hbar)={}_1F_1\left(\tfrac{1}{2}(\mu_l+1)-\frac{\hbar k^2}{4\mu\omega},\mu_l+1,\mu\omega r^2/\hbar\right). \tag{27}$$

For large r, $F(\mu\omega r^2/\hbar)$ would diverge as $\exp(\mu\omega r^2/\hbar)$, thus preventing normalization of the wavefunction. The wavefunction can only be normalized by choosing

$$n_r=\frac{\hbar k^2}{4\mu\omega}-\frac{1}{2}(\mu_l+1) \qquad (n_r=0,1,2,\ldots) \tag{28}$$

which gives the energy eigenvalues (10) as we expect.

Finally, we obtain the energy eigenfunction as

$$\psi_{n_r,l}(r,\varphi)=C_{n_r,l}r^{\mu_l}\exp(-\mu\omega r^2/2\hbar)\sin(l\varphi/a)\,{}_1F_1(-n_r,\mu_l+1;\mu\omega r^2/\hbar) \tag{29}$$

with the normalization constant defined by

$$C_{n_r,l}=\left[(a\pi/2)\int_0^\infty r^{2\mu_l+1}\exp(-\mu\omega r^2/\hbar)\,{}_1F_1^2(-n_r,\mu_l+1;\mu\omega r^2/\hbar)\,dr\right]^{-1/2} \tag{30}$$

which has been found as (12) in section 2.

4.2. Inverse square potential

In polar coordinates, the Schrödinger equation is of the form

$$-\frac{\hbar^2}{2\mu}\left[\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\varphi^2}\right]\psi(r,\varphi)+\frac{g}{r^2}\psi(r,\varphi)=E\psi(r,\varphi). \tag{31}$$

Again, we have the boundary conditions (20) and thus the angular wavefunction (21). However, the radial wavefunction must satisfy the following differential equation:

$$r^2R''(r)+rR'(r)+[k^2r^2-\mu_l^2]R(r)=0 \tag{32}$$

where $k^2=2\mu E/\hbar^2$, $\mu_l^2=2\mu g/\hbar^2+(l/a)^2$. Now, by using the variable $\xi=kr$, (32) reduces to

$$\xi^2\frac{d^2R(\xi)}{d\xi^2}+\xi\frac{dR(\xi)}{d\xi}+(\xi^2-\mu_l^2)R(\xi)=0 \tag{33}$$

whose solution regular at $r=0$ is the Bessel function, or

$$R(\xi)=AJ_{\mu_l}(\xi). \tag{34}$$

Therefore, we have the energy eigenfunction as

$$\psi_{l,E}(r,\varphi)=A\left(\frac{2}{a\pi}\right)^{1/2}J_{\mu_l}(kr)\sin(l\varphi/a) \tag{35}$$

with the normalization constant $A=(\mu/\hbar^2)^{1/2}$ by comparing (17). Comparing the energy eigenvalues and eigenfunctions in sections 2 and 3, we conclude that the propagator (5) is correct.

5. Classical paths for free particle interacting with the rational wedge

Unfortunately, the propagator (5) or (18) cannot be expressed as the sum over 'classical paths'. In order to do so, we restrict ourselves to the case of $\omega=g=0$ (free particle) and $a=n/m$ (rational wedge) and the propagator then reduces to [2, 4, 6]

$$\begin{aligned}K^0_{n/m}&(r'',\varphi'',t;r',\varphi')\\&=\left[\frac{\mu m}{2n\hbar\pi i t}\right]\sum_{k=1}^{m}\left\{\left[1+\sum_{l=1}^{n-1}D^{nl}_{mk}(\varphi''-\varphi')\right]\right.\\&\quad\left.\times\exp\left[\frac{i}{\hbar}S^n_{mk}(\varphi''-\varphi')\right]-(\varphi'\to-\varphi')\right\}.\end{aligned} \tag{36}$$

Hereafter, we use the symbol $(\varphi'\to-\varphi')$ to represent all the terms inside the curly bracket with φ' substituted by $-\varphi'$. In equation (36), we have

$$S^n_{mk}(\varphi''\mp\varphi')=\frac{\mu}{2t}[(r''^2+r'^2)-2r'r''\cos(\varphi''\mp\varphi'+2\pi kn/m)] \tag{37}$$

which will be shown later to be the classical action and the diffractive coefficients

$$\begin{aligned}D^{nl}_{mk}(\varphi''\mp\varphi')&=(i)^{l/n+3}\cos\left[\frac{n-l}{n}(\varphi''\mp\varphi'+2\pi kn/m)\right]\\&\quad\times\int_0^z[J_{-l/n}(u)-(-1)^{l/n}J_{l/n}(u)]\\&\quad\times\exp[iu\cos(\varphi''\mp\varphi'+2\pi kn/m)]\,du\end{aligned} \tag{38}$$

with $z=\mu r'r''/\hbar t$.

From now on, we define the 'classical path' as one uniform motion which starts from the initial position of the particle or its images and arrives at the final position (r'',φ'') of the particle during the time interval t. Using the cosine law of a triangle, it can easily be shown that (37) does represent the classical action on an n-sheeted

Riemann surface. As we see, there are $2m$ classical paths in total. In Cartesian coordinates (37) becomes

$$S^n_{mk}(\varphi''\mp\varphi')=\frac{\mu}{2t}[x''^2+y''^2+x'^2+y'^2+2\sin(2\pi kn/m)(x'y''\mp x''y') \\ -2\cos(2\pi kn/m)(x'x''\pm y'y'')] \tag{39}$$

and

$$\left|\det\frac{\partial^2 S^n_{mk}(\varphi''\mp\varphi')}{\partial \boldsymbol{r}'\partial \boldsymbol{r}''}\right|^{1/2}=\frac{\mu}{t} \tag{40}$$

with the help of (39) and (40), we rewrite (36) in the following form:

$$K^0_{n/m}(\boldsymbol{r}'',\varphi'',t;\boldsymbol{r}',\varphi') \\ =\sum_{k=1}^{m}\left\{\frac{1}{2\pi i\hbar}\left|\det\frac{\partial^2 S^n_{mk}(\varphi''-\varphi')}{\partial \boldsymbol{r}'\partial \boldsymbol{r}''}\right|^{1/2}F^{nl}_{mk}(\varphi''-\varphi') \\ \times\exp\left[\frac{i}{\hbar}S^n_{mk}(\varphi''-\varphi')\right]-(\varphi'\to-\varphi')\right\} \tag{41}$$

which satisfies the van Vleck formula [16] with the extra factor

$$F^{nl}_{mk}(\varphi''\pm\varphi')=\frac{m}{n}[1+D^{nl}_{mk}(\varphi''\mp\varphi')]. \tag{42}$$

Physically, the above factor is difficult to interpret even with the diffractive theory of Keller [17]. However, we notice that each 'classical path' will contribute one non-diffractive term and $n-1$ diffractive terms to the propagator and the factor $F^{nl}_{mk}(\varphi''\pm\varphi')$ can be interpreted as the relative amplitude of the contribution for the 'classical path'.

In order to evaluate (42) explicitly, we consider the case of $n=2$. Substituting

$$J_{1/2}(u)=\sqrt{2/\pi u}\,\sin(u) \qquad J_{-1/2}(u)=\sqrt{2/\pi u}\,\cos(u) \tag{43}$$

into (38), we obtain [6]†

$$F^{21}_{mk}(\varphi''\mp\varphi') \\ =\frac{m}{2}\left(1-2i\left(\frac{\mu r''r'}{\pi\hbar t}\right)^{1/2}\cot[(\varphi''\mp\varphi'+4\pi k/m)/2] \\ \times{}_1F_1\left(\frac{1}{2},\frac{3}{2};\frac{2\mu r'r''\sin^2[(\varphi''\mp\varphi'+4\pi k/m)/2]}{i\hbar t}\right)\right) \tag{44}$$

and

$$F^{21}_{mn}(0)=\frac{m}{2}\left[1-2i\left(\frac{2\mu r'r''}{\pi\hbar t}\right)^{1/2}\right]. \tag{45}$$

† In [6], equation (25) should be corrected as

$$F^{(m,2)}_{1,k}=\frac{m}{2}\left\{1-2i\left(\frac{\lambda r'r''}{\pi\sin(\omega t)}\right)^{1/2}\cot\left[\frac{(\varphi''\mp\varphi'+4\pi k/m}{2}\right] \\ \times{}_1F_1\left(\frac{1}{2},\frac{3}{2};\frac{2\lambda r'r''\sin^2[(\varphi''\mp\varphi'+4\pi k/m)/2]}{i\sin(\omega t)}\right)\right\}.$$

Finally, we have the propagator [3, 6]

$$K_2^0(r'', \varphi'', t; r', \varphi') = \frac{\mu}{4\pi i\hbar t}\left\{\left[1-2i\left(\frac{2\mu r''r'}{\pi\hbar t}\right)^{1/2}\cot[(\varphi''-\varphi')/2] \times {}_1F_1\left(\frac{1}{2},\frac{3}{2};\frac{2\mu r'r''\sin^2[(\varphi''-\varphi')/2]}{i\hbar t}\right)\right]\exp\left[\frac{i\mu}{2\hbar t}[(r'^2+r''^2) - 2r'r''\cos(\varphi''-\varphi')] - (\varphi'\to-\varphi')\right]\right\} \qquad (\varphi''\neq\varphi') \tag{46}$$

and

$$K_2^0(r'', \varphi, t; r', \varphi) = \frac{\mu}{4\pi i\hbar t}\left\{\left[1-2i\left(\frac{2\mu r''r'}{\pi\hbar t}\right)^{1/2}\right]\exp[i\mu(r''-r')^2/2\hbar t] - \left[1-2i\left(\frac{\mu r'r''}{\pi\hbar t}\right)\cot(\varphi)\,{}_1F_1\left(\frac{1}{2},\frac{3}{2};\frac{2\mu r'r''\sin^2(\varphi)}{i\hbar t}\right)\right] \times\exp\left[\frac{i\mu}{2\hbar t}[(r'^2+r''^2)-2r'r''\cos(2\varphi)]\right]\right\} \tag{47}$$

for the half-barrier case ($m=1$ and $n=2$) [3]. Here we should mention that most of the results in this section have already been obtained in [6] for the more general case of $\omega\neq 0$. However, up to now we are still not able to show that (21) in [6] does represent the classical action due to the complicated classical paths connected the initial point and the final point in a harmonic oscillator case. That is the main reason why we repeat all the formulae and interpret them correctly for the free particle case.

Once we determine the initial position of the particle, the configuration space can be divided into three different regions for the final position of the particle. As shown in figure 2, there exist two real classical paths in region I, one real and one imaginary classical path in region II and two imaginary classical paths in region III (shadow region), respectively. Finally, we should remark that once the classical path (real or imaginary) reflects from the surfaces of the wedge, its contribution to the propagator (35) will change signs or multiply by -1, as we expect.

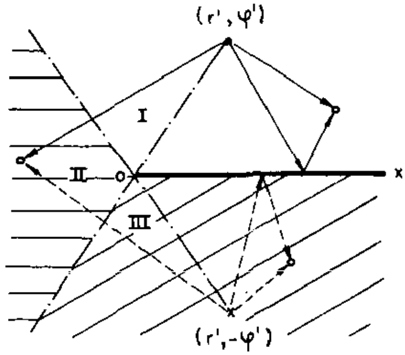

Figure 2. The classical paths in different regions. Full lines represent the real classical paths and dotted line the imaginary classical paths.

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